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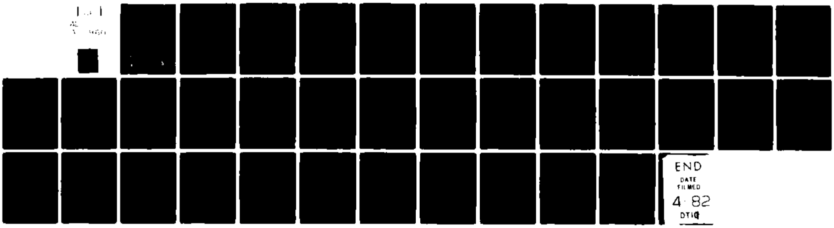

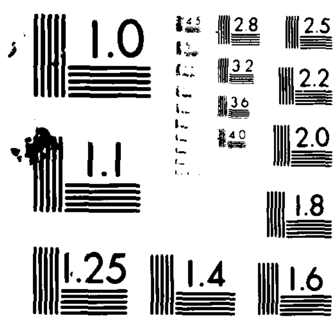

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SUMMARY

The simple line-vortex model of laterally-symmetric separated flow past a circular cone at incidence, originally proposed by Bryson, is studied in detail, both numerically and analytically through asymptotic expansions. Two branches of solutions are found, an 'upper' branch which includes Bryson's original results, and a lower branch, on which the solutions are found to be physically unrealistic. The physically realistic branch is shown to yield multiple solutions for separation lines to windward of a critical position. A proposed modification to the condition imposed on the flow at the separation line is shown to have little effect on the behaviour of the model.

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1 INTRODUCTION

There is considerable interest in the aerodynamics of aircraft and missiles at high angle of attack. The flow past the nose of such a vehicle will often exhibit vortex separation. This separation usually starts at the pointed apex, and hence considerable information might be obtained from studying flows past cones. As these flows involve separation one ought, in principle, to solve the full Navier-Stokes equations. However, Fiddes[1] has shown that an inviscid model involving vortex sheets can give useful predictions of real flows involving separation from smooth surfaces, when the separation line is determined by a calculation in which the interaction with the viscous flow is represented.

Bryson[2] introduced a very simple inviscid model for the flow past a slender circular cone at incidence. He made the slender-body assumption and modelled the separated flow by isolated line-vortices. Each such vortex is linked by a cut to the separation line from which, in the real flow, it would be fed. It is required that the net force on the vortex and cut should vanish. The position of the separation line must be specified, and Bryson required that this line should be a streamline of the three-dimensional flow.

Although this model is too simple to be of practical use, its properties are of interest because on the one hand it represents a degenerate case of the vortex-sheet model and on the other hand its solutions may provide initial approximations for the numerical procedures needed to calculate the behaviour of the vortex-sheet model. Its simplicity allows a complete investigation of its solutions.

The study described in section 2 reveals a 'lower' branch of solutions, in addition to the upper branch which contains Bryson's original results. On this lower branch the vortex lies on the windward side of the 'separation line'. It is shown that, for the solutions on this branch, the postulated 'separation line' is actually an attachment line, so that these solutions must be discarded. It emerges that, even on the physically realistic branch, multiple solutions are possible if the separation line lies to windward of a critical position, which is itself to windward of the position Bryson chose for his numerical investigation. Physical arguments are put forward for preferring one of these solutions.

The condition that the separation line should be a streamline of the three-dimensional flow is somewhat arbitrary. Smith[3] has proposed an alternative condition which is consistent with the vortex-sheet model of the flow. In section 3, the effect on the solution of using the alternative condition is shown to be small. The main features revealed in section 2 are unchanged.

2 BRYSON'S MODEL

For completeness, the model of Bryson[2] is re-derived in section 2.1, before the solution techniques and results are described in the remainder of section 2.

2.1 Equations

Some of the notation to be used is illustrated in Fig 1. A Cartesian coordinate system is adopted, and the apex of the cone is assumed to lie at the origin of the

coordinate system. The axis of the cone is taken to lie along Ox, and the coordinates in any plane perpendicular to this axis are y and z.

The slender-body assumption is made. Hence the velocity in a plane perpendicular to the axis of the body (the cross-flow plane) may be described by a potential which satisfies Laplace's equation in the cross-flow plane, and so is the real part of a complex analytic function. The equations of the model will therefore be formulated in terms of a complex variable, $\omega = y + iz$. In the slender-body description of the attached flow past a conical body, the component of the velocity in the cross-flow plane is conical, in the sense of being constant along rays through the apex. It will be assumed that the separated flow is also conical; in particular, that the separation line is straight, and that the line-vortices which represent the real separation vortices are also rectilinear.

s, the radius of the circular cross-section of the cone in the cross-flow plane, is equal to $x \tan\gamma$, where γ is the (small) semi-angle of the cone. The separation lines are $\omega = \omega_s s$ where $\omega_s = e^{i\theta}$ or $-e^{-i\theta}$, and θ, which satisfies $-90^\circ \leqslant \theta^\circ \leqslant 90^\circ$, is termed the separation angle. $\omega_v s$ and $-\bar{\omega}_v s$ (where a bar denotes the complex conjugate) are the positions of the vortices modelling the symmetric separation. The velocity in the far field is $(U\cos\alpha,\ 0,\ U\sin\alpha)$ where α is the (small) angle of incidence in radians, and U is the free stream speed. Because the flow is conical the circulation of the vortices is proportional to x and is taken to be equal to $2\pi\Gamma s U \tan\gamma$.

The condition that the component of the velocity normal to the surface of the cone is zero gives

$$v_n = U\tan\gamma ,$$

where v_n is the velocity component on the circle along its outward normal in the (y,z) plane. The complex potential, W, can be seen to be:

$$W = U\tan\gamma\, s\ln\omega - i\alpha\left(\omega - \frac{s^2}{\omega}\right)U + \frac{\Gamma s U\tan\gamma}{i}\left[\ln(\omega - \omega_v s) - \ln\left(\omega - \frac{s}{\bar{\omega}_v}\right) - \ln(\omega + s\bar{\omega}_v) + \ln\left(\omega + \frac{s}{\omega_v}\right)\right]. \qquad (2\text{-}1)$$

The first term represents the effect of the streamwise growth of the cross-section, the second term describes the attached flow round the cross-section of the cone, and the third term gives the effects of the two vortices and their images in the circle. The condition which Bryson[2] imposes on the flow at the separation line is that the velocity there is parallel to the separation line. In the real viscous flow, the limiting direction of the flow, as the wall is approached, is parallel to the separation line. For an inviscid, vortex-sheet model of the separated flow, Smith[3] points out that the flow on the downstream side (but not on the upstream side) of the separation line must be parallel to it. An alternative condition valid for both vortex-sheet and line-vortex models is explored in section 3. Bryson's condition leads to:

$$i\alpha\left(1+\frac{s^2}{\omega^2}\right) = \frac{\Gamma s \tan\gamma}{i}\left[\frac{1}{\omega-\omega_v s} - \frac{1}{\omega-(s/\bar{\omega}_v)} - \frac{1}{\omega+s\bar{\omega}_v} + \frac{1}{\omega+(s/\omega_v)}\right] \tag{2-2}$$

at $\omega = \omega_s s = se^{i\theta}$. This equation is satisfied in a trivial sense for $\omega = \pm is$. Otherwise it implies:

$$\Gamma \tan\gamma = \frac{\alpha\left(1+\bar{\omega}_v e^{-i\theta}\right)\left(1+\omega_v e^{i\theta}\right)\left(1-\omega_v e^{-i\theta}\right)\left(1-\bar{\omega}_v e^{i\theta}\right)}{(\omega_v\bar{\omega}_v-1)(\omega_v+\bar{\omega}_v)} \quad . \tag{2-3}$$

Bryson assumes that the line vortex is inclined at a small angle to the local velocity vector so that the resulting force on the vortex exactly balances the force on the cut. For the starboard vortex, this leads to:

$$(\omega_v s - \omega_s s)\frac{d}{dx}(2\pi\Gamma sU\tan\gamma) + (2\pi\Gamma sU\tan\gamma)\frac{d}{dx}(\omega_v s) - (2\pi\Gamma s\tan\gamma)(v_v + iw_v) = 0 \ , \tag{2-4}$$

where the first term comes from the force on the cut, the second term from the force on the vortex due to the free-stream and the final term from the force on the vortex due to the cross-flow. $v_v + iw_v$ is the cross-flow velocity at the vortex at $\omega_v s$, and is given by

$$v_v - iw_v = \left.\frac{dW_1}{d\omega}\right|_{\omega=s\omega_v} \tag{2-5}$$

where $W_1 = W - \frac{\Gamma sU\tan\gamma}{i}\ln(\omega-\omega_v s)$, the derivative of W_1 , rather than W , being required because a straight line-vortex has no self-induced velocity.

Substituting equation (2-5) into the complex conjugate of equation (2-4) and recalling that $s = x\tan\gamma$ leads to

$$\tan\gamma\left(2\bar{\omega}_v - e^{-i\theta}\right) = -i\alpha\left(1+\frac{1}{\omega_v^2}\right) + \frac{\tan\gamma}{\omega_v} + \frac{\Gamma\tan\gamma}{i}\left[-\frac{1}{\omega_v-(1/\bar{\omega}_v)} - \frac{1}{\omega_v+\bar{\omega}_v} + \frac{1}{\omega_v+(1/\omega_v)}\right]$$

so that

$$a^{-1}\left(2\bar{\omega}_v - e^{-i\theta} - \frac{1}{\omega_v}\right) = -i\left(1+\frac{1}{\omega_v^2}\right) + \frac{\Gamma a^{-1} i}{\omega_v+\bar{\omega}_v} + \frac{\Gamma a^{-1} i(\omega_v+\bar{\omega}_v)}{\left(\omega_v^2+1\right)(\omega_v\bar{\omega}_v-1)} \tag{2-6}$$

where $a = \alpha/\tan\gamma$, and is termed the incidence parameter.

Substituting Γ from equation (2-3) into this gives

$$a^{-1}\left(2\bar{\omega}_v - e^{-i\theta} - \frac{1}{\omega_v}\right) = -i\left(1 + \frac{1}{\omega_v^2}\right) + \frac{i\left(1 + \bar{\omega}_v e^{-i\theta}\right)\left(1 + \omega_v e^{i\theta}\right)\left(1 - \omega_v e^{-i\theta}\right)\left(1 - \bar{\omega}_v e^{i\theta}\right)}{(\omega_v\bar{\omega}_v - 1)(\omega_v + \bar{\omega}_v)^2}$$

$$+ \frac{i\left(1 + \bar{\omega}_v e^{-i\theta}\right)\left(1 + \omega_v e^{i\theta}\right)\left(1 - \omega_v e^{-i\theta}\right)\left(1 - \bar{\omega}_v e^{i\theta}\right)}{\left(\omega_v^2 + 1\right)(\omega_v\bar{\omega}_v - 1)^2} \; . \qquad (2\text{-}7)$$

If $t = \omega_v e^{-i\theta}$ this becomes

$$a^{-1}e^{-i\theta}\left(2\bar{t} - 1 - \frac{1}{t}\right) = -i\left(1 + \frac{e^{-2i\theta}}{t^2}\right) + \frac{i(1 + \bar{t}e^{-2i\theta})(1 + te^{2i\theta})(1 - t)(1 - \bar{t})}{(t\bar{t} - 1)(te^{i\theta} + \bar{t}e^{-i\theta})^2}$$

$$+ \frac{i(1 + \bar{t}e^{-2i\theta})(1 + te^{2i\theta})(1 - t)(1 - \bar{t})}{(1 + t^2e^{2i\theta})(1 + \bar{t}^2e^{-2i\theta})(t\bar{t} - 1)^2}\,(1 + \bar{t}^2e^{-2i\theta}) \; .$$

$$\ldots\ldots \quad (2\text{-}8)$$

Equations (2-7) and (2-8) are each equivalent to two real equations relating the incidence parameter, a, and the real and imaginary parts of the complex vortex position, ω_v or t respectively. It is convenient to proceed by specifying the imaginary part of ω_v or t and then solving to find the incidence parameter and the real part of ω_v or t respectively. From a knowledge of these three parameters, Γ can be evaluated.

The final equation needed is an expression for the normal force. Let C_N be the normal force on the cone forward of the plane x = constant, non-dimensionalised with the cross-sectional area of the cone ($= \pi s^2$) and $\frac{1}{2}\rho U^2$ where U is, as before, the free-stream velocity and ρ is the density of the fluid. Then it can be shown that[2]

$$\frac{C_N}{\tan\gamma} = 2a + 8\Gamma y_v\left(1 - \frac{1}{R^2}\right) \qquad (2\text{-}9)$$

where $\omega_v = y_v + iz_v$ and $|\omega_v| = R$. C_N can be found if the vortex position and circulation are known.

2.2 Numerical methods

It is possible to apply three criteria to assess the relevance of solutions of equations (2-7) and (2-8). It is assumed that the y and z coordinate directions are chosen so that α is positive. Hence one restriction must be

$$a > 0 \; . \qquad (2\text{-}10a)$$

The vortices modelling the separation must lie outside the cone (and their images inside the cone), and so

$$|\omega_v| > 1 \; . \qquad (2\text{-}10b)$$

If the cut is regarded as the vestige of a vortex sheet, it is necessary to ensure that the two cuts joining the vortices to their respective separation lines do not cross. For symmetric separation (in which the two cuts are reflections of each other in the line $y = 0$) it is therefore necessary that the y coordinate of the vortex and its associated separation point have the same sign. This leads to

$$\mathcal{R}(\omega_v) \geqslant 0 \quad , \tag{2-10c}$$

because $-90^\circ < \theta < 90^\circ$.

First the solution of equation (2-7) will be considered. It will be assumed that $\mathcal{I}(\omega_v)$ is specified. If real and imaginary parts of equation (2-7) are taken, two equations relating a^{-1} and $\mathcal{R}(\omega_v)$ result. If a^{-1} is eliminated between these two equations a 17th-order polynomial for $y_v = \mathcal{R}(\omega_v)$ can be obtained. Taking $z_v = \mathcal{I}(\omega_v)$ and $R^2 = y_v^2 + z_v^2$, the polynomial can be written as

$$\begin{aligned} & 2y_v z_v\left(-2z_v R^2 + \sin\theta\, R^2 + z_v\right) 4y_v^2 \times \\ & \quad \times \left[R^4\left(1+\bar{\omega}_v e^{-i\theta}\right)\left(1+\omega_v e^{i\theta}\right)\left(1-\omega_v e^{-i\theta}\right)\left(1-\bar{\omega}_v e^{i\theta}\right) - \left(R^2-1\right)^2\left(\omega_v^2+1\right)\left(\bar{\omega}_v^2+1\right)\right] \\ & = \\ & \left(2y_v R^2 - R^2\cos\theta - y_v\right) \times \\ & \quad \times \Big[-4y_v^2 R^4\left(R^2-1\right)^2\left(\omega_v^2+1\right)\left(\bar{\omega}_v^2+1\right) - \left(y_v^2-z_v^2\right)\left(R^2-1\right)^2\left(\omega_v^2+1\right)\left(\bar{\omega}_v^2+1\right)4y_v^2 \\ & \qquad + R^4\left(R^2-1\right)\left(\omega_v^2+1\right)\left(\bar{\omega}_v^2+1\right)\left(1+\bar{\omega}_v e^{-i\theta}\right)\left(1+\omega_v e^{i\theta}\right)\left(1-\omega_v e^{-i\theta}\right)\left(1-\bar{\omega}_v e^{i\theta}\right) \\ & \qquad + R^4 4y_v^2\left(1+\bar{\omega}_v e^{-i\theta}\right)\left(1+\omega_v e^{i\theta}\right)\left(1-\omega_v e^{-i\theta}\right)\left(1-\bar{\omega}_v e^{i\theta}\right)\left(y_v^2-z_v^2+1\right)\Big] \quad . \end{aligned}$$

(Despite its appearance this expression is real and can be written as a polynomial in y_v . It has been left in this form for brevity.)

All roots of this were found using the method of Grant and Hitchins as implemented by the Numerical Algorithms Group[4] in their subroutine CØ2AEF. Unacceptable solutions for $\mathcal{R}(\omega_v)$ were then eliminated using the requirement that they must be real (the polynomial may have some complex zeros, even although all the coefficients are real), and the criteria of equations (2-10).

The method seems to work well in general, but was poor near the surface of the cone. This is due to the occurrence of repeated roots of the polynomial at the surface, because the subroutine, although second-order convergent for single roots, exhibits numerical first-order convergence for repeated roots.

When the method just described for the solution of equation (2-7) was unsatisfactory, equation (2-8) was solved (using FORTRAN with double-precision real variables) for specified $\mathcal{I}(t)$, to find just one root, in the following way. By taking real and imaginary parts of equation (2-8) two expressions for a^{-1} can be found. Subtracting these two expressions gives a function ψ, say, of $\xi = \mathcal{R}(t)$, the zeros of which give the roots of equation (2-8). To find a zero a modified interpolation scheme was used, in which two values of ξ must be supplied, say ξ_A and ξ_B, such that the signs of $\psi(\xi_A)$ and $\psi(\xi_B)$ differ. Then there is at least one root of $\psi = 0$ between ξ_A and ξ_B and the scheme finds one root. Since the method* does not seem to be well known it is explained in Fig 2.

Although the scheme is similar to an ordinary interpolation scheme, which does not involve ordinate halving (*ie* the instructions $F_B = F_B/2$ and $F_A = F_A/2$ would be omitted from the bottom two boxes in Fig 2b), experience has shown that the modified scheme is often much faster and rarely much slower than the ordinary scheme. It has the additional advantage that any root can be found to a known accuracy, because the iterations are, in general, stopped when $|\xi_A - \xi_B|$ is sufficiently small, and the error in the zero has an upper bound of $|\xi_A - \xi_B|$. In the ordinary scheme the root is said to have been found when $\psi(\xi)$ is sufficiently small. As $|\xi_A - \xi_B|$ may still be large it is difficult to estimate the accuracy to which the root has been found.

2.3 Asymptotic solution when the vortex is near the surface of the cone

In this section equation (2-8) will be studied, rather than equation (2-7), because the asymptotic analysis appears to be slightly simpler. A similar analysis could have been carried out on equation (2-7).

The small parameter is taken as $\delta = \mathcal{I}(t)$. Details of the asymptotic analysis are given in Appendix A. Here only the results are given.

$$\mathcal{R}(t) = 1 + A\delta + \left(\frac{8A \tan\theta - 4}{9}\right)\delta^2 + O(\delta^3) \quad ,$$

$$a^{-1} = \frac{2\sin\theta}{3} + \left(\frac{2\cos\theta}{3} - \frac{10A\sin\theta}{9}\right)\delta + O(\delta^2) \quad , \qquad (2\text{-}11)$$

where $A = 1/\sqrt{3}$ if $\delta > 0$
$A = -1/\sqrt{3}$ if $\delta < 0$,

so that $A\delta > 0$. Hence

$$a = \frac{3\,\mathrm{cosec}\,\theta}{2}\left[1 + \left(\frac{5A}{3} - \cot\theta\right)\delta + O(\delta^2)\right] \quad , \qquad (2\text{-}12)$$

$$\Gamma = 4a\cos\theta(A\sigma) + O(\delta^2) \quad , \qquad (2\text{-}13)$$

* The method of solution was suggested to the author by Dr F.A. Johnson, while at RRE Malvern.

from equation (2-3), and

$$\frac{C_N}{\tan\gamma} = 2a + O(\delta^2) \quad , \tag{2-14}$$

from equation (2-9). These results are in good agreement with the numerical results for sufficiently small $|\delta|$.

For given θ , the locus of possible vortex positions (a varying along the locus) appears to have two branches near the surface of the cone. Their asymptotic form, to $O(\delta)$, is shown by the dashed lines in Fig 3, for the two cases $\theta = 56^\circ$ (the case studied by Bryson[2]) and $\theta = 14.3^\circ$ ($= 0.25$ radian), the actual loci being shown by the dotted curves. Also noted on Fig 3 is the behaviour (as the vortex moves away from the surface of the cone) of the incidence parameter, the circulation and the normal force. The critical value of θ, θ_c , which determines the change of behaviour of Γ with a on the upper branch, can be found from equation (2-11). It is given by

$$\tan\theta_c = \left(\frac{2}{3}\right)\left(\frac{9}{10A}\right) = \frac{3\sqrt{3}}{5} \quad ,$$

as $A = +1/\sqrt{3}$ on the upper branch, so that $\theta_c = 46.1^\circ$. A similar change of behaviour of Γ with a can be observed on the lower branch, the critical θ here being $-\theta_c$.

In order to clarify the nature of the upper and lower branches of the solution, it is helpful to examine the flow near the separation line. The complex conjugate of the cross-flow velocity is given by the derivative of (2-1):

$$v - iw = \frac{dW}{d\omega} = U\tan\gamma\left[\frac{s}{\omega} - ia\left(1 + \frac{s^2}{\omega^2}\right) + \frac{\Gamma s}{i}\left(\frac{1}{\omega - \omega_v s} - \frac{\bar{\omega}_v}{\omega\bar{\omega}_v - s} - \frac{1}{\omega + s\bar{\omega}_v} + \frac{\omega_v}{\omega\omega_v + s}\right)\right] \quad . \tag{2-15}$$

As in (2-8), the position of the starboard vortex is written as $\omega_v = te^{i\theta}$, where, by (2-11)

$$t = 1 + \delta\left(i \pm \frac{1}{\sqrt{3}}\right) + O(\delta^2)$$

and the upper and lower signs refer to the upper and lower branches, on which δ is positive and negative, respectively. The velocity is required at points which are closer to the separation line than is the vortex, so we write

$$\omega = s(1 + \zeta)e^{i\theta} \quad ,$$

where $|\zeta| \ll |\delta|$.

With $\Gamma = (4/\sqrt{3})a|\delta|\cos\theta$, from (2-13), we find the leading terms of (2-15) are

$$v - iw = U\tan\gamma\, e^{-i\theta}\left(1 + \frac{3a\zeta\cos\theta}{\delta} + \ldots\right) \quad . \tag{2-16}$$

At a point in the fluid on the normal to the surface of the cone, ζ is real and positive. The velocity along the outward normal to the surface of the cone is given by a contribution $(v \cos\theta + w \sin\theta) \cos\gamma$ from the cross-flow, together with a component $-U \sin\gamma(1 + O(\zeta))$ from the free stream, *ie* by (2-16):

$$\frac{3aU\zeta \sin\gamma \cos\theta}{\delta} ,$$

to leading order in ζ/δ. The flow is therefore away from the surface for $\delta > 0$ and towards the surface for $\delta < 0$. Hence the 'separation line' can represent a genuine separation line for the upper branch of the solution, but represents an attachment line for the lower branch of the solution. The lower branch must therefore be discarded as physically irrelevant.

2.4 Numerical results

It has already been noted that two branches of the locus of vortex positions for fixed θ, only the upper one of which corresponds to separation at θ, are obtained from the asymptotic solution when the vortex is close to the cone. The two branches obtained from the solution of equation (2-7) by the method of section 2.2 are shown by the dashed curves in Fig 4 for $\theta = 56^\circ$ (the case studied by Bryson[2]). Whilst only the upper branch is physically relevant, for completeness results for both branches are shown in Fig 4 and in subsequent figures.

By solving either equation (2-7) or equation (2-8), Figs 5 and 6 can be obtained. These two figures show loci of possible vortex positions with either θ varying and a constant, or a varying and θ constant. In Fig 5 attention is restricted to the upper branches (at constant θ), while in Fig 6 attention is restricted to the lower branches. All the solutions for $a > 0$ lie in the upper quadrant, as shown. If $a < 0$, the mirror image in the horizontal axis of Fig 5 or 6 is obtained. If a locus for constant θ, say $\theta = -14.3^\circ$, in Fig 5 is followed into the lower quadrant, it continues smoothly as the mirror image of the locus for $\theta = 14.3^\circ$ in Fig 6, *ie* as a locus of lower branch solutions. Conversely, the lower branch loci of Fig 6 continue as the mirror images of the upper branch loci of Fig 5.

The possible non-uniqueness of solutions of the Bryson model is clearly illustrated. If attention is restricted to the lower branches, Fig 6, then, for given $\theta > 0$, there is a unique solution for the vortex position if $a \geq \frac{3}{2} \operatorname{cosec}\theta$, and no solution if $a < \frac{3}{2} \operatorname{cosec}\theta$. However, for the same θ and a ($> \frac{3}{2} \operatorname{cosec}\theta$), there will be at least one upper branch solution, Fig 5, for the vortex position, and all the upper branch solutions will differ from the lower branch solution. For the upper branch (Fig 5), corresponding to physically realistic solutions, the existence of solutions now depends on θ and a in a more complicated way, which is illustrated in Fig 7. A curve $a = a_L(\theta)$ represents the lowest value of a for which a solution with separation at θ exists. If $\theta > \theta_c \simeq 46.1^\circ$, then $a_L(\theta) \equiv \frac{3}{2} \operatorname{cosec}\theta$, the same boundary as was found for the existence of lower branch solutions, and there is a unique solution for $a \geq a_L(\theta)$, with no solution for $a < a_L(\theta)$. If $0 < \theta < \theta_c$, $a_L(\theta)$ is a numerically-defined

function, with $a_L(\theta) < \frac{1}{2}$ cosec θ , and there are three possibilities: for $a < a_L(\theta)$, there is no solution; for $a_L(\theta) < a \leqslant \frac{1}{2}$ cosec θ , there are two solutions, which coincide for $a = a_L(\theta)$; and for $a > \frac{1}{2}$ cosec θ , there is just one solution. If $\theta \leqslant 0$, there is no solution for $a < a_L(\theta)$ and there are two solutions for $a \geqslant a_L(\theta)$, coinciding when $a = a_L(\theta)$.

Figs 5 and 6 also illustrate the regions in which the vortex can lie. In Fig 5 the locus of vortex positions for $\theta = -90^{\circ}$ gives one of the boundaries, because of the requirement that the two (symmetric) cuts must not cross. This locus, and the upper branches of the loci of vortex positions for fixed θ with $-90^{\circ} < \theta \leqslant 0^{\circ}$ are all asymptotically inclined at $\pi/3$ to the horizontal. Their approach to this asymptote is described in Appendix B. Another boundary is formed by the Föppl curve. This is the locus of points at which vortices can lie symmetrically in the wake of a circular cylinder in plane flow (see page 223 of Ref 5, for instance), and so corresponds naturally to the limit of $a \to \infty$. Its equation is:

$$z^2 = 1 + y^2 + 2y\sqrt{1 + y^2}$$

so it, too, is asymptotically inclined at $\pi/3$ to the horizontal. The remaining boundary (apart from the surface of the cone) consists of part of the line $z = 0$. The intersection of this line with the locus of vortex positions for $\theta = -90^{\circ}$ may be found by substituting $\omega_v = y$ and $e^{i\theta} = -i$ into equation (2-7). Taking the real part of the resulting equation gives $a^{-1} = 0$. Taking the imaginary part, with $a^{-1} = 0$, gives a quartic in y , which is a quadratic in y^2 . This may be solved to give

$$y = \left(2 + \left(\frac{7}{3}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} = 1.878 \quad .$$

In Fig 6 the lower branch of the locus of the vortex positions when $\theta = 90^{\circ}$ and the line $z = 0$ give two of the boundaries. By the same method as above these can be shown to intersect at

$$y = 1.878 \qquad z = 0 \ .$$

In Figs 8 and 9 the variation of the circulation parameter Γ with the incidence parameter, a , at constant separation position, θ , is illustrated. The abscissa starts at $a = 1.5$ because there is no solution to the Bryson model when $a < 1.5$. In Fig 8 the upper branches are considered, while in Fig 9 attention is restricted to the lower branches. The change in character of the curves in Fig 8 occurs at $\theta = \theta_c \simeq 46.1^{\circ}$. For $0 < \theta < \theta_c$, there are two values of Γ , corresponding to the two solutions indicated in Fig 7, for $a_L(\theta) \leqslant a < \frac{1}{2}$ cosec θ , *eg* for $4.17 \leqslant a < 6.06$ when $\theta = 14.3^{\circ}$. For $\theta > \theta_c$, Γ is uniquely determined (on the upper branch) by θ and a .

3 MODEL WITH MODIFIED BOUNDARY CONDITION ON THE SEPARATION LINE

3.1 Equations

Bryson's model requires the separation line to be a streamline of the three-dimensional flow, so that v_{t_m} , the (mean) tangential cross-flow velocity component at

the separation line, is zero. *However by considering* the behaviour of a vortex-sheet model at the separation line, Smith[3] has derived a condition on v_{t_m}, which can be applied whether the sheet is present or not. For a body of revolution, such as a circular cone, with the separation line along a meridian, Smith's condition becomes

$$v_{t_m} = \sqrt{\frac{U}{2}\frac{d}{dx}\,(2\pi\Gamma sU\tan\gamma)}$$

in the direction of increasing θ, using the same notation as in section 2.1.

If this is the only modification made to the model, it may be seen from the potential given in equation (2-1) that, at the separation point in the $y-z$ plane, $\omega = \omega_s s = se^{i\theta}$,

$$-i\alpha\left(1+\frac{s^2}{\omega^2}\right)U + \frac{\Gamma sU\tan\gamma}{i}\left[\frac{1}{\omega-\omega_v s} - \frac{1}{\omega-(s/\bar{\omega}_v)} - \frac{1}{\omega+s\bar{\omega}_v} + \frac{1}{\omega+(s/\omega_v)}\right] = -U\tan\gamma\sqrt{\pi\Gamma}\,ie^{-i\theta}$$

instead of equation (2-2), where the positive square root is to be taken. This gives, instead of equation (2-3),

$$\frac{\Gamma(\omega_v\bar{\omega}_v-1)\;2\cos\theta(\omega_v+\bar{\omega}_v)}{\left(1+\bar{\omega}_v e^{-i\theta}\right)\left(1+\omega_v e^{i\theta}\right)\left(1-\omega_v e^{-i\theta}\right)\left(1-\bar{\omega}_v e^{i\theta}\right)} - 2a\cos\theta + \sqrt{\pi\Gamma} = 0\,. \qquad (3\text{-}1)$$

Equation (2-6), obtained from the requirement that the net force on the vortex and cut is zero, is unaltered.

If $t = \omega_v e^{-i\theta}$, equation (3-1) becomes

$$\frac{\Gamma(t\bar{t}-1)\;2\cos\theta(te^{i\theta}+\bar{t}e^{-i\theta})}{(1+\bar{t}e^{-2i\theta})(1+te^{2i\theta})(1-t)(1-\bar{t})} - 2a\cos\theta + \sqrt{\pi\Gamma} = 0 \qquad (3\text{-}2)$$

and equation (2-6) may be written in the form

$$1 = \frac{-ia\left(1+\dfrac{e^{-2i\theta}}{t^2}\right) + \dfrac{i\Gamma}{(te^{i\theta}+\bar{t}e^{-i\theta})} + \dfrac{i\Gamma(te^{i\theta}+\bar{t}e^{-i\theta})}{(t\bar{t}-1)(1+t^2e^{2i\theta})}}{e^{-i\theta}\left(2\bar{t}-1-\dfrac{1}{t}\right)}\,. \qquad (3\text{-}3)$$

The normal force is still given by equation (2-9).

If the imaginary part of t is specified, equations (3-2) and (3-3) may be solved to find Γ, a and the real part of t. From a knowledge of these the normal force may be calculated.

3.2 Numerical methods

The method of solution of equations (3-2) and (3-3) is similar to that used to solve equation (2-8).

The three criteria of relevance (equations (2-10) of section 2.2) still apply and for the same reasons. It is convenient to specify $\mathscr{I}(t)$.

For any guessed $\mathscr{R}(t)$, Γ/a is evaluated from the imaginary part of equation (3-3) and then a determined from the real part of this equation. These values are substituted into the expression on the left-hand side of equation (3-2), taking the positive square root. The modified interpolation scheme described in section 2.2 and Fig 2 is then employed to find $\mathscr{R}(t)$ so that equation (3-2) is satisfied.

This method does not have the advantage of ensuring that all possible solutions of the equations are found, but a thorough search failed to reveal any relevant solutions other than those referred to in sections 3.3 and 3.4.

3.3 Asymptotic solution when the vortex is near the surface of the cone

As in section 2.3, the small parameter is $\delta = \mathscr{I}(t)$. Details of the asymptotic analysis are given in Appendix C. Here the results are given.

$$\mathscr{R}(t) = 1 + A\delta - \frac{2A\delta}{3}\sqrt{\frac{2\pi A\delta \tan\theta}{3}} + \left(\frac{8A\tan\theta}{9} + \frac{4\pi\tan\theta}{81} - \frac{4}{9}\right)\delta^2 + 0\left(\delta^{\frac{5}{2}}\right)$$

$$a^{-1} = \frac{2\sin\theta}{3} + \left(\frac{2\cos\theta}{3} - \frac{10A\sin\theta}{9}\right)\delta + 0\left(\delta^{\frac{3}{2}}\right)$$

$$\Gamma a^{-1} = 4A\delta\cos\theta - \frac{8A\delta}{3}\sqrt{\frac{\pi A\delta\sin 2\theta}{3}} + \left(\frac{16\pi\sin\theta}{31} - \frac{16A\sin\theta}{9} - \frac{16\cos\theta}{9}\right)\delta^2 + 0\left(\delta^{\frac{5}{2}}\right)$$

where $A = +1/\sqrt{3}$ if $\delta > 0$,
$A = -1/\sqrt{3}$ if $\delta < 0$,

so that $A\delta > 0$. Hence

$$a = \frac{3\operatorname{cosec}\theta}{2}\left[1 + \left(\frac{5A}{3} - \cot\theta\right)\delta + 0\left(\delta^{\frac{3}{2}}\right)\right]$$

and

$$\frac{C_N}{\tan\gamma} = 2a + 0\left(\delta^2\right) \quad .$$

These results (which are only applicable when $\theta > 0$ because of the restriction $a > 0$) agree to $0(\delta)$ with the results given in section 2.3, but the next term is of different form.

For given θ the locus of possible vortex positions (a varying along the locus) again has two branches near the surface of the cone.

The leading term of the solution is again shown by the dashed lines in Fig 3, but the dotted curves no longer give the actual loci of the vortex. The critical value of θ, θ_c , is still

$$\tan^{-1}\left(\frac{3\sqrt{3}}{5}\right) = 46.1^\circ \quad .$$

The previous analysis of the velocity near the point $\omega = se^{i\theta}$ holds, and it follows that the lower branch solutions must again be discarded.

3.4 Numerical results

Again it was found that, because of the restriction $a > 0$, all solutions satisfied $\mathscr{I}(\omega_v) \geqslant 0$. For constant θ the locus of possible vortex positions still has two branches, as is illustrated by the solid curves in Fig 4, for the case $\theta = 56^o$. Fig 4 also illustrates the effect of modifying the condition at the separation line.

In Fig 10 the variation of the circulation parameter, Γ, with the incidence parameter, a, for $\theta = 56^o$ is illustrated by the solid curves. This figure also shows the effect of modifying the condition at the separation line.

As the effect of modifying the boundary condition is quantitative rather than qualitative all the comments of section 2.4 are still valid. The critical θ, θ_c, is still $\tan^{-1}\left(3\sqrt{3}/5\right)$ and it is still possible to have non-unique solutions.

In his example III.1, Nangia[6] also considers Smith's[3] boundary condition. Fig 4 here and Fig 8 of Nangia agree to within the accuracy of curve plotting.

4 DISCUSSION

Bryson's model for separated flow round a circular cone has been investigated, and the occurrence of multiple solutions clarified. For fixed separation position, θ, a branch of the locus of the vortex position not previously reported has been found. However it has been shown that for this branch (termed the 'lower' branch in this paper) the so-called separation line is in fact an attachment line. Hence this branch of the solution can be discarded on physical grounds.

Then, for $\theta \geqslant \theta_c$ $\left(\theta_c = \tan^{-1}\left(3\sqrt{3}/5\right) = 46.1^o\right)$, there is a unique solution for any given θ and incidence parameter, a, provided $a \geqslant \frac{1}{2}$ cosec θ. If $a < \frac{1}{2}$ cosec θ there is no solution. The incidence parameter and the strength of the vortex increase as the position of the vortex moves away from the surface of the cone. It is for a value of θ (56^o) in this range that Bryson[2] calculated solutions of his model, finding the realistic, upper branch of the present investigation. For $0 < \theta < \theta_c$ with $a_L(\theta) < a \leqslant \frac{1}{2}$ cosec θ, and for $-90^o \leqslant \theta \leqslant 0^o$ with $a_L(\theta) < a$ there are two possible vortex positions and strengths, so, as a increases from $a_L(\theta)$ with θ fixed, there are two branches of the solution. Along one of them the vortex moves away from the cone and becomes stronger as a increases. If $0 < \theta < \theta_c$, this branch is a smooth continuation of the unique solution which exists for $a > \frac{1}{2}$ cosec θ. It approaches the Föppl curve as $a \to \infty$. Along the other branch the vortex moves towards the cone and becomes weaker as a increases. This is physically unrealistic. For $\theta > 0$, the surface of the cone is reached when $a = \frac{1}{2}$ cosec θ.

In a real flow, the separation position, θ, must depend on the incidence parameter, a, through the viscous mechanism governing separation. It is likely that, as a increases, separation first occurs near $\theta = \pi/2$, and that θ falls as a increases. By the time θ has fallen as far as θ_c the vortex will be well away from the surface, so that, when θ falls below θ_c, the vortex will be on the physically realistic branch

referred to above. In terms of the bifurcation locus $a_L(\theta)$ and the separation position $\theta(a)$ in the viscous flow, it is expected that $a_L(\theta(a))$ is appreciably less than a itself, so that there is no way in which the non-physical branch can be reached.

Smith's[3] modified condition on the separation line has been shown to have a small quantitative, but no qualitative, effect on the results. Hence this modification has no effect on any of the comments made in this section.

Appendix A

ASYMPTOTIC SOLUTION OF BRYSON MODEL WHEN THE VORTEX IS NEAR THE SURFACE OF THE CONE

The equation to be solved is

$$a^{-1}e^{-i\theta}\left(2\bar{t} - 1 - \frac{1}{t}\right) = -i\left(1 + \frac{e^{-2i\theta}}{t^2}\right) + \frac{i(1+\bar{t}e^{-2i\theta})(1+te^{2i\theta})(1-t)(1-\bar{t})}{(t\bar{t}-1)(te^{i\theta}+\bar{t}e^{-i\theta})^2}$$

$$+ \frac{i(1+\bar{t}e^{-2i\theta})(1+te^{2i\theta})(1-t)(1-\bar{t})(1+\bar{t}^2e^{-2i\theta})}{(1+t^2e^{2i\theta})(1+\bar{t}^2e^{-2i\theta})(t\bar{t}-1)^2} \qquad \text{(A-1)}$$

which is equation (2-8), given in section 2.1.

Let the small parameter be $\delta = \mathcal{I}(t)$ and let $\mathcal{R}(t) = 1 + q$ where q is small. It can be shown that acceptable solutions can only be obtained when $q = O(\delta)$. Hence let

$$q = A\delta + O(\delta^2)$$

where A is a constant to be determined. To lowest order the left-hand side of equation (A-1) is then

$$a^{-1}e^{-i\theta}(3A - i)\delta$$

and the right-hand side is

$$-i(1+e^{-2i\theta}) + \frac{i(A^2+1)}{4A^2}(1+e^{-2i\theta}) \quad .$$

By comparing these, it may be seen that, since the left-hand side is $O(\delta)$, $A^2 = 1/3$. As

$$|t|^2 = 1 + 2A\delta + O(\delta^2) \quad \text{and} \quad |t| \geqslant 1 \ ,$$

where $A = +1/\sqrt{3}$ if $\delta > 0$,
and $A = -1/\sqrt{3}$ if $\delta < 0$.

Now let

$$\mathcal{R}(t) = 1 + A\delta + B\delta^2 + O(\delta^3) \quad ,$$

multiply equation (A-1) by $e^{i\theta}$, and expand it to order δ. Taking real parts gives

$$a^{-1} = \frac{2\sin\theta}{3} + O(\delta) \quad .$$

Then, by considering the imaginary part, it may be shown that

$$B = \frac{(8A\tan\theta - 4)}{9} \quad .$$

Now, writing

$$a^{-1} = \frac{2\sin\theta}{3} + D\delta + O(\delta^2)$$

and expanding the real part of $e^{i\theta}$ times equation (A-1) *to order* δ^2 it is found that

$$D = \frac{2\cos\theta}{3} - \frac{10A\sin\theta}{9} .$$

Thus the asymptotic expansions given in section 2.3 are obtained, equations (2-3) and (2-9) of section 2.2 being used to obtain the asymptotic forms of Γ and C_N .

Appendix B

ASYMPTOTIC FORM OF THE UPPER BRANCH OF THE LOCUS OF VORTEX POSITIONS WHEN THE SEPARATION ANGLE, θ, IS $\leqslant 0°$, FAR FROM THE CONE

Equation (2-7), given in section 2.1, is

$$a^{-1}\left(2\bar{\omega}_v - e^{-i\theta} - \frac{1}{\omega_v}\right) = -i\left(1 + \frac{1}{\omega_v^2}\right) + \frac{i\left(1+\bar{\omega}_v e^{-i\theta}\right)\left(1+\omega_v e^{i\theta}\right)\left(1-\omega_v e^{-i\theta}\right)\left(1-\bar{\omega}_v e^{i\theta}\right)}{(\omega_v\bar{\omega}_v - 1)(\omega_v + \bar{\omega}_v)^2}$$

$$+ \frac{i\left(1+\bar{\omega}_v e^{-i\theta}\right)\left(1+\omega_v e^{i\theta}\right)\left(1-\omega_v e^{-i\theta}\right)\left(1-\bar{\omega}_v e^{i\theta}\right)}{\left(\omega_v^2+1\right)(\omega_v\bar{\omega}_v - 1)^2} \quad . \qquad \text{(B-1)}$$

If $|\omega_v|$ is assumed to be large the left-hand side of this equation, to lowest order, is

$$2a^{-1}\bar{\omega}_v$$

and the right-hand side is, to lowest order,

$$i\left(\frac{\omega_v\bar{\omega}_v}{(\omega_v+\bar{\omega}_v)^2} - 1\right) \quad .$$

As the right-hand side is, to this order, purely imaginary

$$a^{-1}\bar{\omega}_v = o(1)$$

and $\omega_v = \rho e^{i\pi/3}$ to lowest order, where ρ, the modulus of ω_v, cannot be determined at this order. Let $1/\rho$ be the small parameter of the expansion. If

$$\omega_v = \rho\left[e^{i\pi/3} + \frac{Be^{i\phi}}{\rho} + O(\rho^{-2})\right]$$

and equation (B-1) is expanded to order ρ^{-1} it is found that, to this order, the right-hand side of equation (B-1) is purely imaginary. Thus

$$a^{-1}\bar{\omega}_v = o(\rho^{-1}) \quad ,$$

and ρ cannot be determined at this order. ϕ is an arbitrary angle and will be taken equal to $5\pi/6$ as the vectors represented by $e^{i\pi/3}$ and $e^{i5\pi/6}$ are orthogonal. This leads to

$$B = \sin\theta \quad .$$

Now let

$$\omega_v = \rho\left[e^{i\pi/3}\left(1 + \frac{i \sin\theta}{\rho}\right) + \frac{A}{\rho^2} e^{i\psi} + O(\rho^{-3})\right]$$

and expand equation (B-1) to order ρ^{-2} . As the right-hand side is purely imaginary

$$a^{-1}\bar{\omega}_v = o(\rho^{-2}) \quad .$$

Thus ρ cannot be determined at this order and ψ is an arbitrary angle. It will, as above, be taken equal to $5\pi/6$. This gives

$$A = -\frac{1}{\sqrt{3}} \quad .$$

Let

$$\omega_v = \rho\left[e^{i\pi/3}\left(1 + \frac{i \sin\theta}{\rho} - \frac{i}{\sqrt{3}\rho^2}\right) + \frac{Ce^{i\chi}}{\rho^3} + O(\rho^{-4})\right]$$

and expand equation (B-1) to order ρ^{-3} . If $\theta = 0$ it is found that $C = 0$, but if $\theta < 0$ the real part of the resulting equation gives

$$a^{-1} = -3 \sin\theta\, \rho^{-4} \quad ,$$

and hence

$$\rho = (-3a \sin\theta)^{\frac{1}{4}} \quad .$$

The imaginary part of the equation gives a relationship between C and χ , but it is not possible to determine C and χ completely at this order.

For the case $\theta = 0$, let

$$\omega_v = \rho\left[e^{i\pi/3} - \frac{1}{\sqrt{3}\rho^2} e^{i5\pi/6} + \frac{De^{i\gamma}}{\rho^4} + O(\rho^{-5})\right]$$

and expand equation (B-1) to order ρ^{-4} . Then the real part of the resulting equation gives

$$a^{-1} = \frac{2\sqrt{3}}{\rho^5}$$

and hence

$$\rho = \left(2\sqrt{3}a\right)^{\frac{1}{5}} \quad .$$

The imaginary part of the equation will give a relationship between D and γ , but it will not be possible to determine D and γ completely at this order.

Appendix C

ASYMPTOTIC SOLUTION OF THE MODEL WITH MODIFIED BOUNDARY CONDITIONS WHEN THE VORTEX IS NEAR THE SURFACE OF THE CONE

The equations to be solved are

$$\frac{\lambda(t\bar{t}-1)\ 2\cos\theta(te^{i\theta}+\bar{t}e^{-i\theta})}{(1+\bar{t}e^{-2i\theta})(1+te^{2i\theta})(1-t)(1-\bar{t})} - 2\cos\theta + \sqrt{\pi\lambda a^{-1}} = 0 \tag{C-1}$$

and

$$a^{-1}e^{-i\theta}\left(2\bar{t}-1-\frac{1}{t}\right) = -i\left(1+\frac{e^{-2i\theta}}{t^2}\right) + \frac{\lambda i}{te^{i\theta}+\bar{t}e^{-i\theta}} + \frac{\lambda i(te^{i\theta}+\bar{t}e^{-i\theta})}{(1+t^2e^{2i\theta})(t\bar{t}-1)} , \tag{C-2}$$

which are equations (3-2) and (3-3), given in section 3.1, with Γ replaced by λa for convenience.

Let the small parameter be $\delta = \mathcal{I}(t)$ and let $\mathcal{R}(t) = 1 + q$ where q is small. It can be shown that acceptable solutions can only be obtained when q and λ are $O(\delta)$. Let

$$q = A\delta + \text{higher order terms}$$

where A is a constant to be determined. To lowest order, equation (C-1) gives

$$\lambda = (A^2+1)\cos\theta\,\frac{\delta}{A} .$$

To lowest order the left-hand side of equation (C-2) is

$$a^{-1}e^{-i\theta}(3A-i)\delta ,$$

and the right-hand side is

$$-i(1+e^{-2i\theta}) + \frac{i(A^2+1)}{2A^2}\cos\theta\ e^{-i\theta} .$$

By comparing these, it may be seen that

$$a^{-1}(3A-i)\delta = O(\delta)$$

and hence

$$A^2 = \frac{1}{3} .$$

As in Appendix A, the requirement that $|t| > 1$ gives

$$A = +\frac{1}{\sqrt{3}} \qquad \text{if } \delta > 0$$

$$A = -\frac{1}{\sqrt{3}} \qquad \text{if } \delta < 0 .$$

Thus

$$\lambda = 4A \cos\theta\, \delta$$

to lowest order.

Inspection of equation (C-1) now suggests

$$\lambda = 4A \cos\theta\, \delta + C\delta^{\frac{3}{2}} + 0(\delta^2)$$

provided that a^{-1} is $0(1)$. This suggests

$$\mathcal{R}(t) = 1 + A\delta + B\delta^{\frac{3}{2}} + 0(\delta^2) \quad .$$

Equations (C-1) and (C-2) expanded to $0(\delta^{\frac{1}{2}})$ give

$$B\delta^{\frac{1}{2}} = -\frac{A}{3}\sqrt{\frac{4\pi AG\delta}{\cos\theta}} \;, \qquad C = 4B\cos\theta$$

where $a^{-1} = G + 0(\delta^{\frac{1}{2}})$ and G cannot be determined at this order. Letting

$$a^{-1} = G + H\delta^{\frac{1}{2}} + 0(\delta)$$

and expanding the real part of equation (C-2) to order $\delta^{\frac{3}{2}}$ gives

$$G = \frac{2\sin\theta}{3} \;, \qquad H = 0$$

so that the assumption that a^{-1} is $0(1)$ is justified.

Now let

$$\mathcal{R}(t) = 1 + A\delta + B\delta^{\frac{3}{2}} + E\delta^2 + 0\left(\delta^{\frac{5}{2}}\right)$$

$$\lambda = 4A\cos\theta\,\delta + C\delta^{\frac{3}{2}} + F\delta^2 + 0\left(\delta^{\frac{5}{2}}\right)$$

$$a^{-1} = \frac{2\sin\theta}{3} + K\delta + 0\left(\delta^{\frac{3}{2}}\right) \quad .$$

From equation (C-1) and the imaginary part of equation (C-2) to order δ

$$E = \frac{8A\tan\theta}{9} + \frac{4\pi\tan\theta}{81} - \frac{4}{9}$$

$$F = \frac{16\pi\sin\theta}{81} - \frac{16A\sin\theta}{9} - \frac{16\cos\theta}{9} \quad .$$

Then from the real part of equation (C-2) to order δ^2

$$K = \frac{2\cos\theta}{3} - \frac{10A\sin\theta}{9} \quad .$$

Thus the asymptotic expansions given in section 3.3 are obtained, where the equation

$$\lambda = \frac{\Gamma}{a}$$

has been used to find Γ, and equation (2-9) has been used for the normal force.

LIST OF SYMBOLS

A	$= (\operatorname{sgn} \delta)/3^{\frac{1}{2}}$
a	$= \alpha/\tan\gamma$, incidence parameter
$a_L(\theta)$	lowest value of a for which a solution exists for given θ
C_N	coefficient of normal force, based on cross-sectional area
R	$= \lvert\omega_v\rvert$
s	radius of circular cross-section of cone
t	$= \omega_v e^{-i\theta}$
U	free-stream speed
v,w	components of cross-flow velocity
v_n	component of cross-flow velocity normal to section of cone
v_{t_m}	component of cross-flow velocity tangential to section of cone
v_v, w_v	components of cross-flow velocity at position of starboard vortex
W	complex potential
W_1	modified complex potential
x,y,z	Cartesian coordinates, see Fig '
y_v, z_v	coordinates of starboard vortex in cross-flow plane
α	angle of incidence
Γ	(circulation of starboard vortex)$/2\pi Us \tan\gamma$
γ	semi-angle of cone
δ	$= \mathcal{J}(t)$, small parameter in expansions
ζ	small complex quantity, section 2.3
θ	position of separation line, see Fig 1
θ_c	critical value of θ $(\tan^{-1} 3^{\frac{3}{2}}/5)$
ξ	argument of ψ
λ	$= \Gamma/a$
ρ	density of fluid, except in Appendix C
ψ	function of which a zero is sought
ω	$= y + iz$, complex variable in cross-flow plane
ω_s	value of ω/s on starboard separation line
ω_v	value of ω/s at starboard vortex

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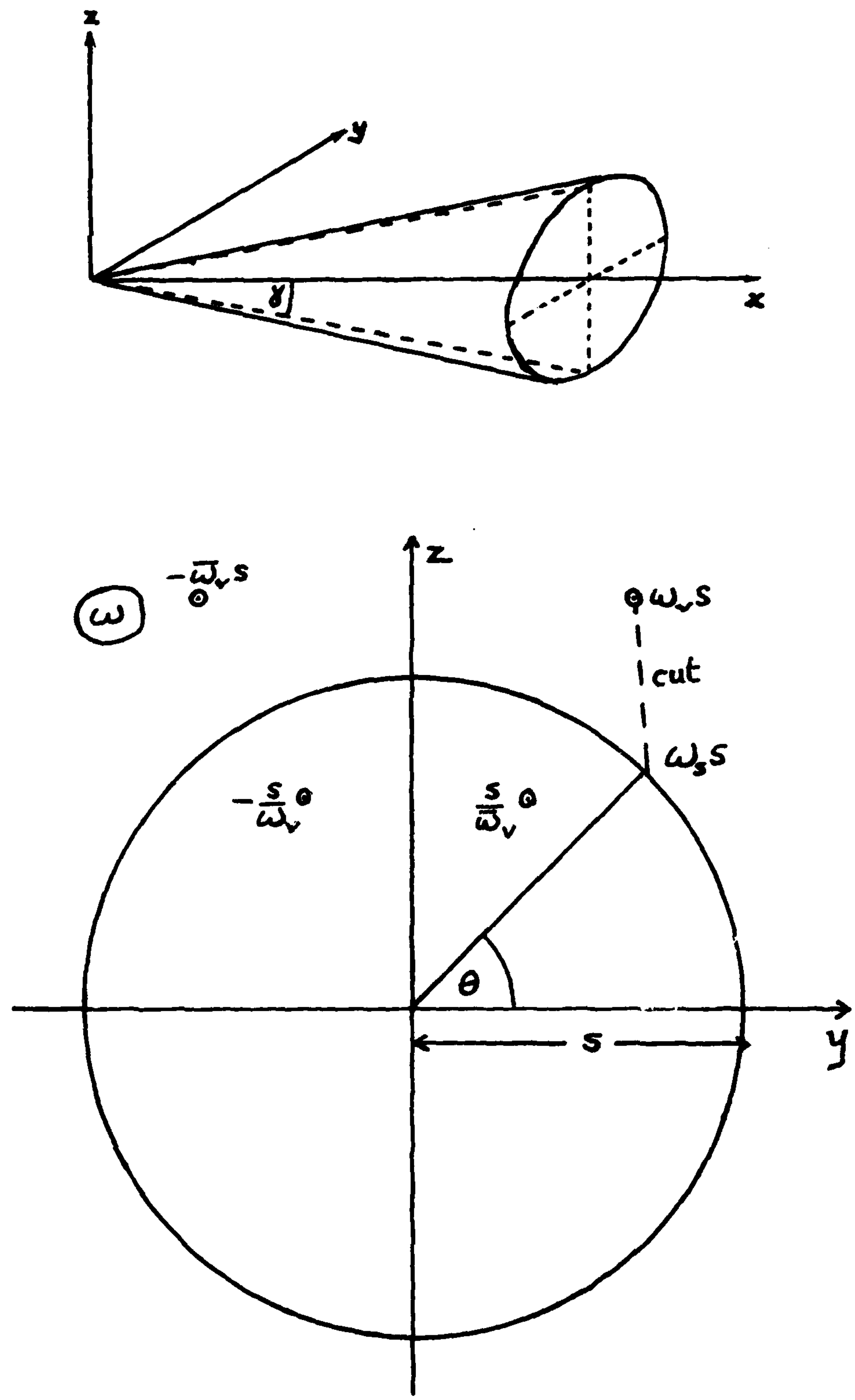

Fig 1 Notation

Fig 2a

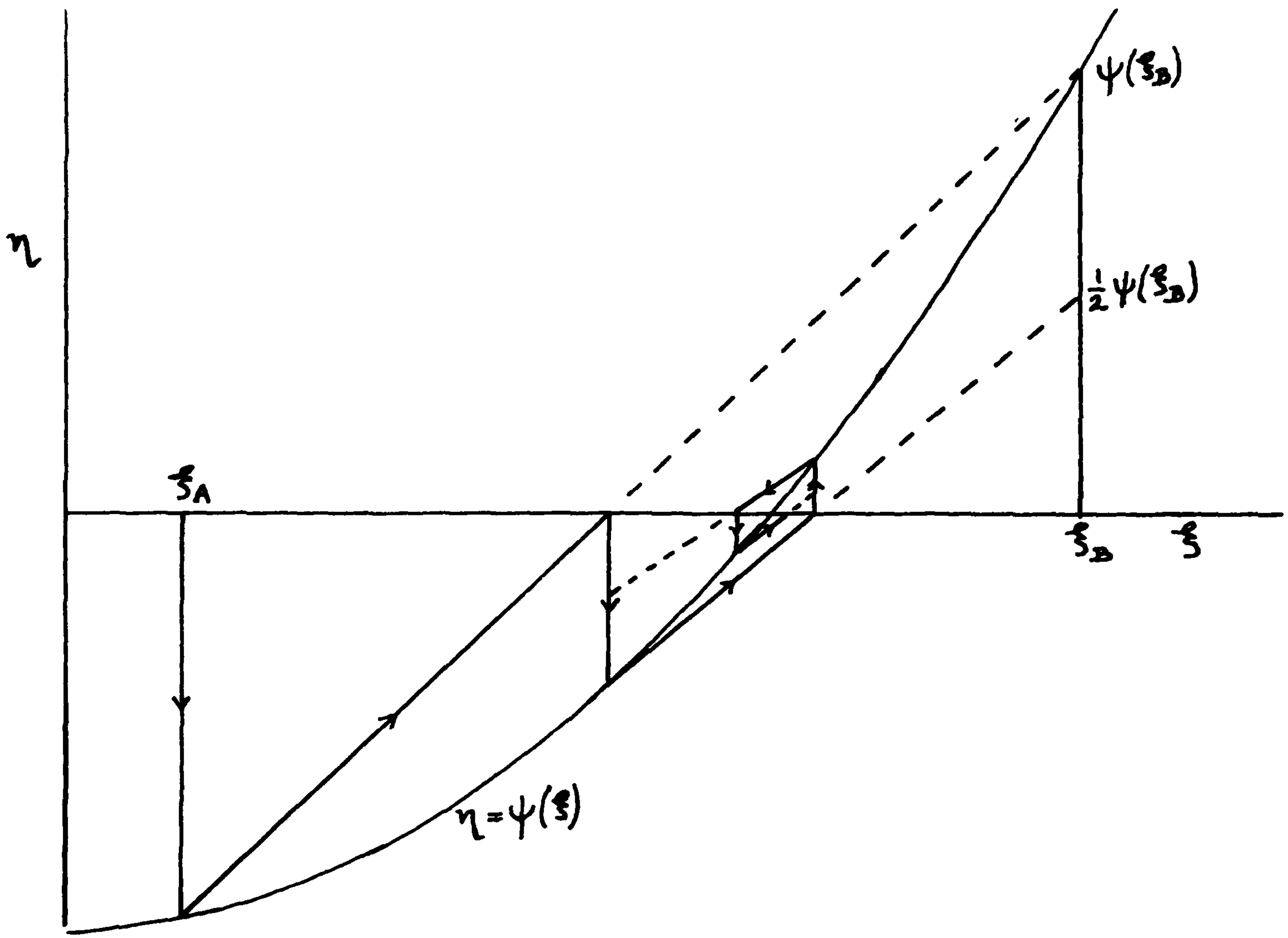

Fig 2 Modified interpolation scheme for finding a root of $\psi(\xi) = 0$
(a) Graphical representation

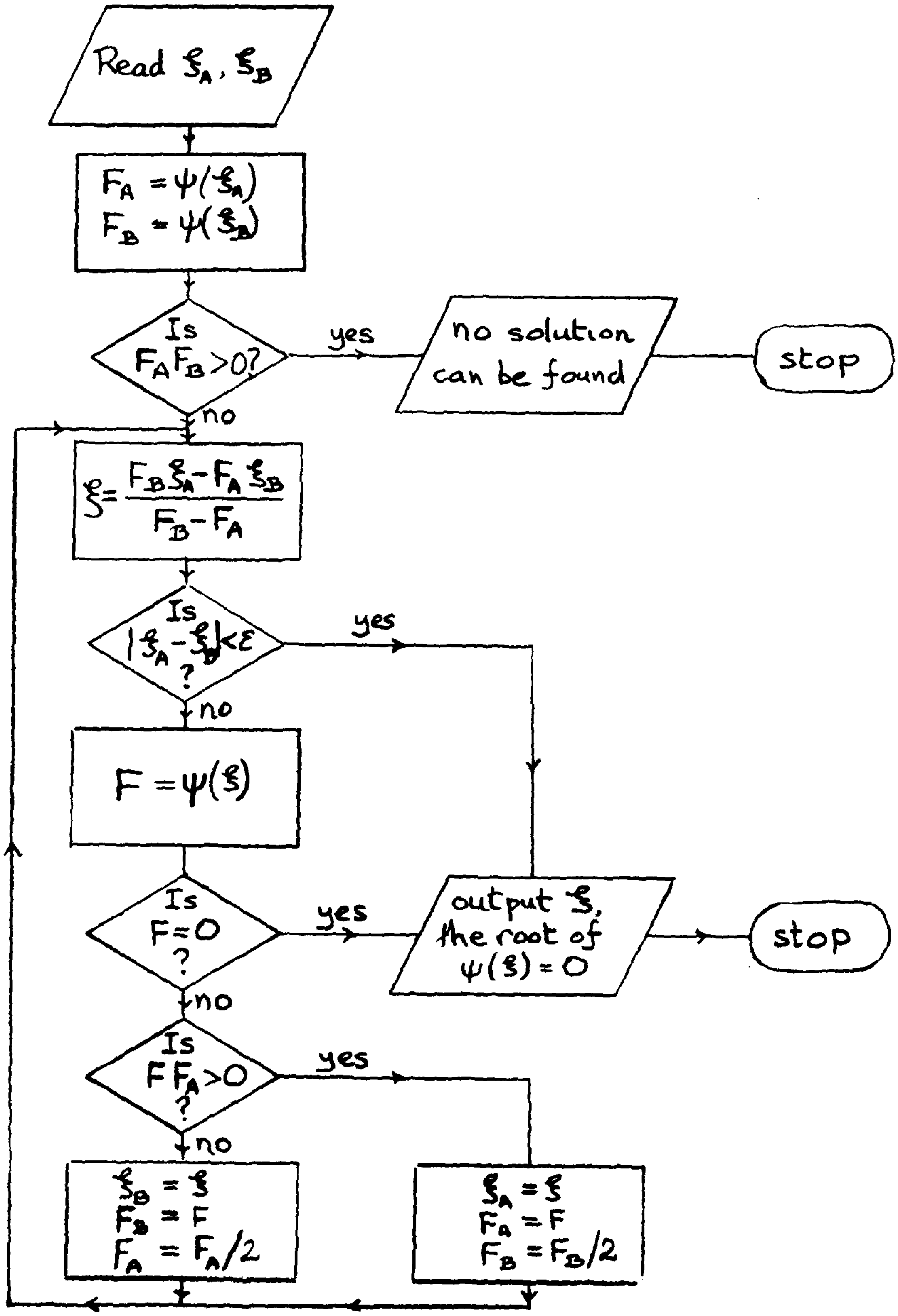

Fig 2 (b) Flowchart

Fig 3

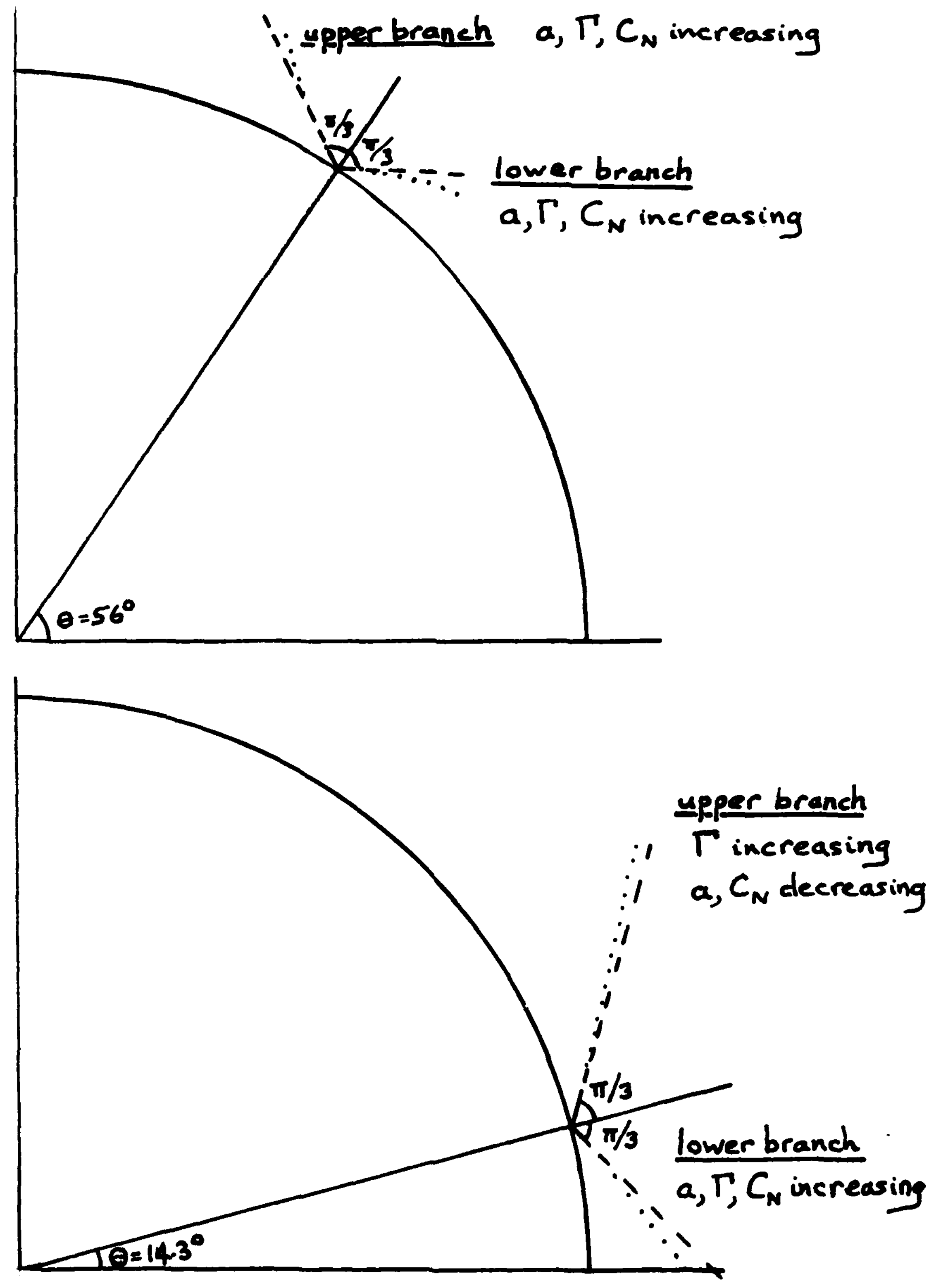

Fig 3 Vortex positions near the cone, Bryson model

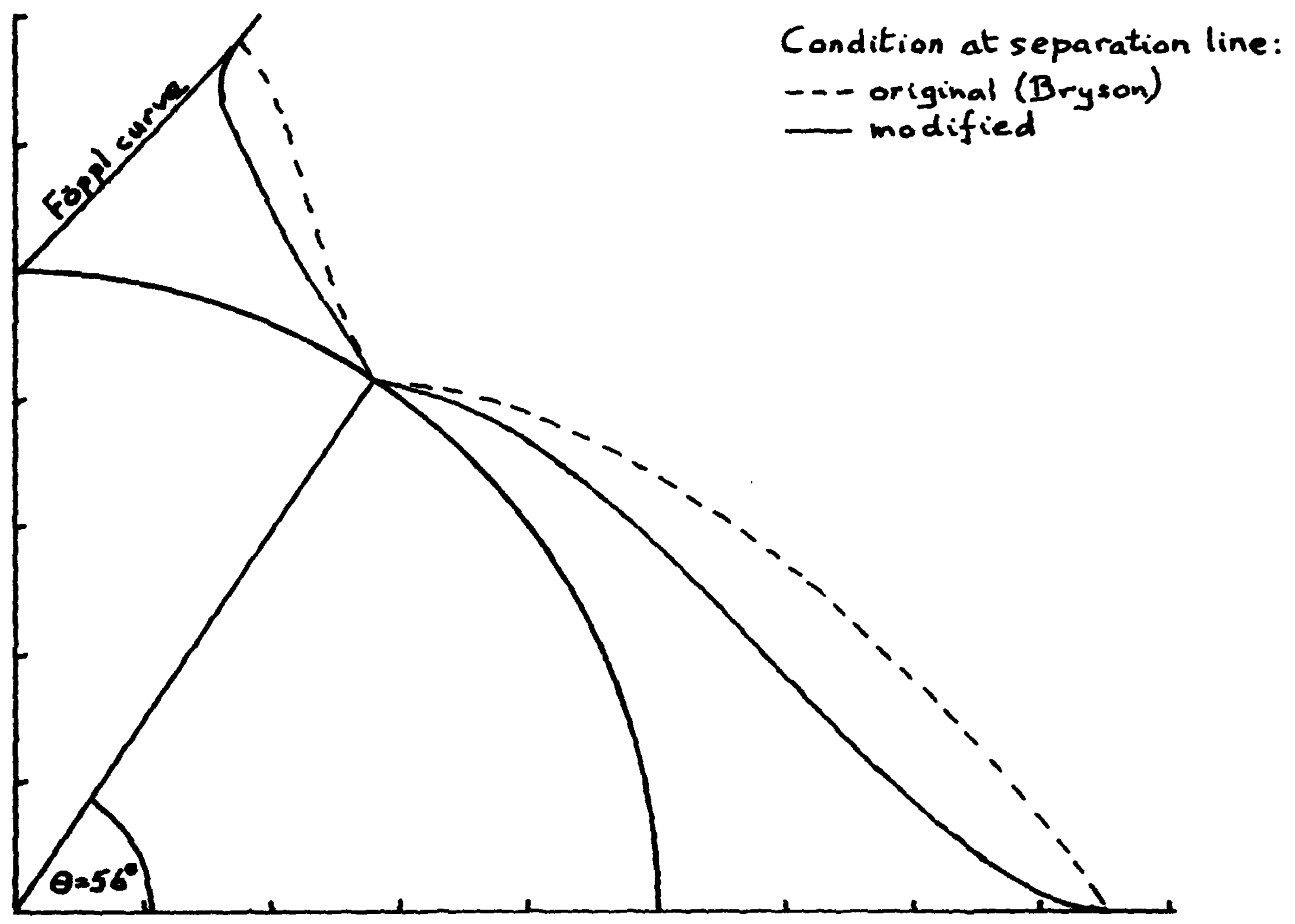

Fig 4 Vortex positions for $\theta = 56^{\circ}$, both models

Fig 5

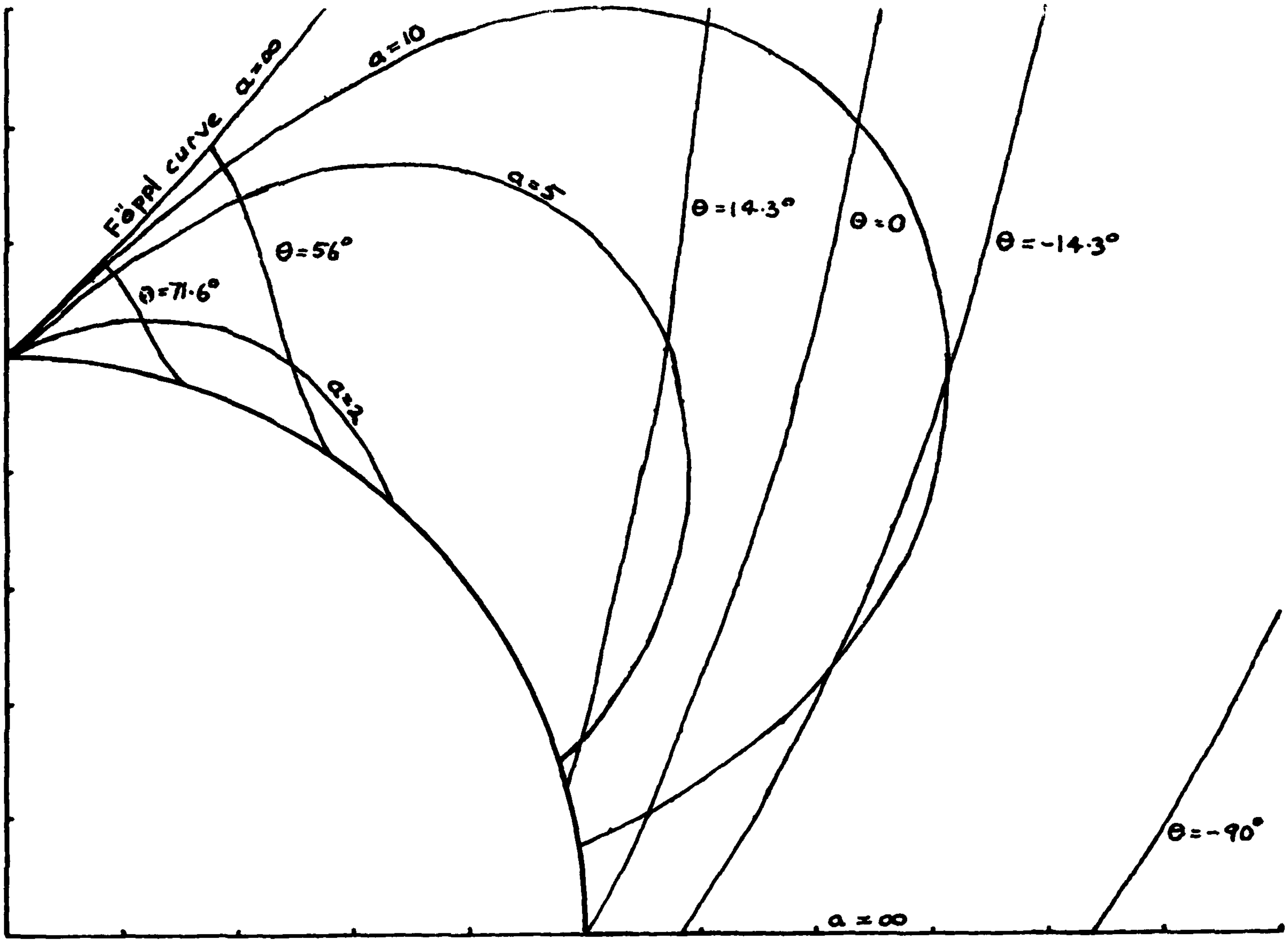

Fig 5 Loci of vortex positions for varying θ and a , Bryson model, upper branch

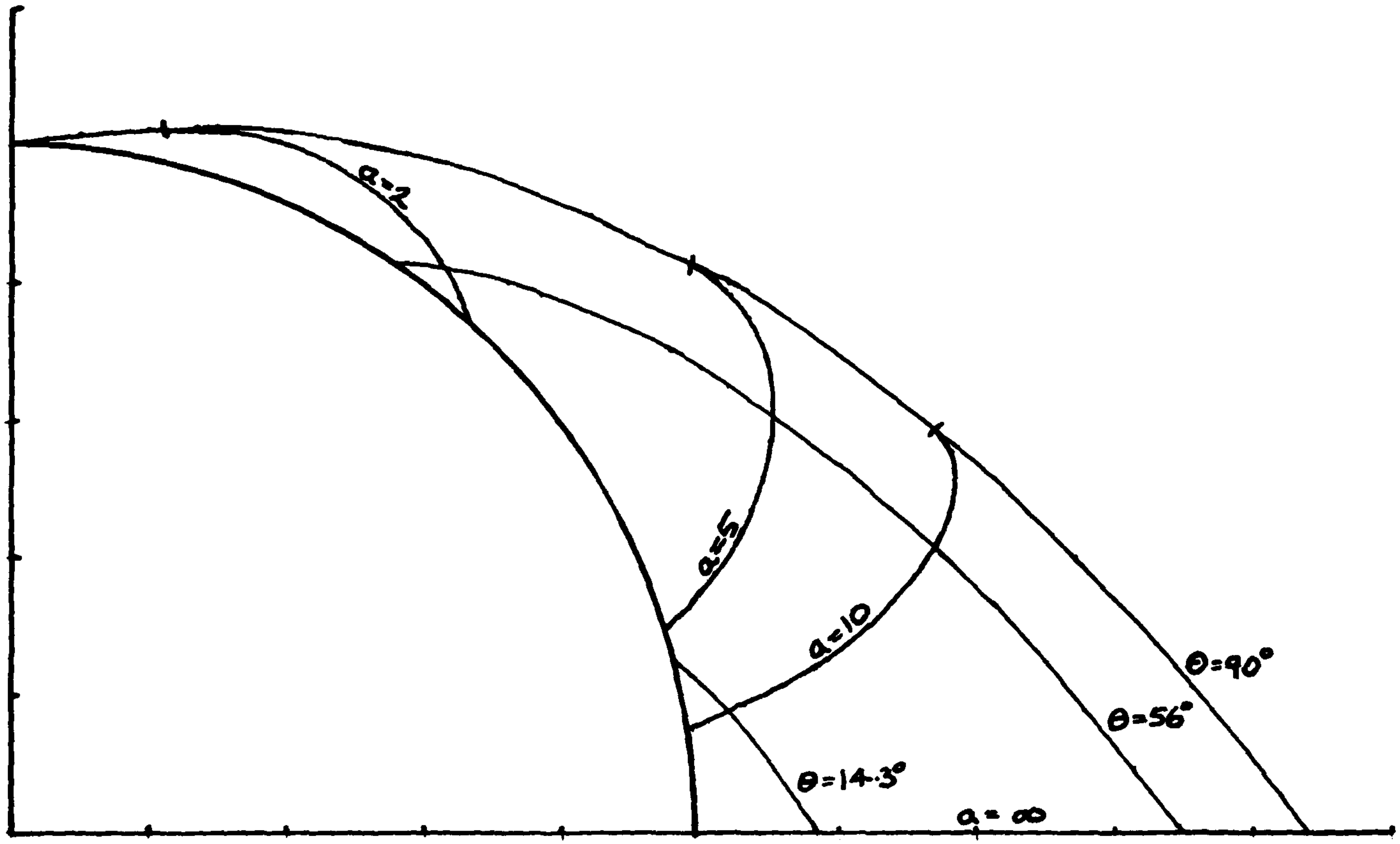

Fig 6 Loci of vortex positions for varying θ and a , Bryson model, lower branch

Fig 7

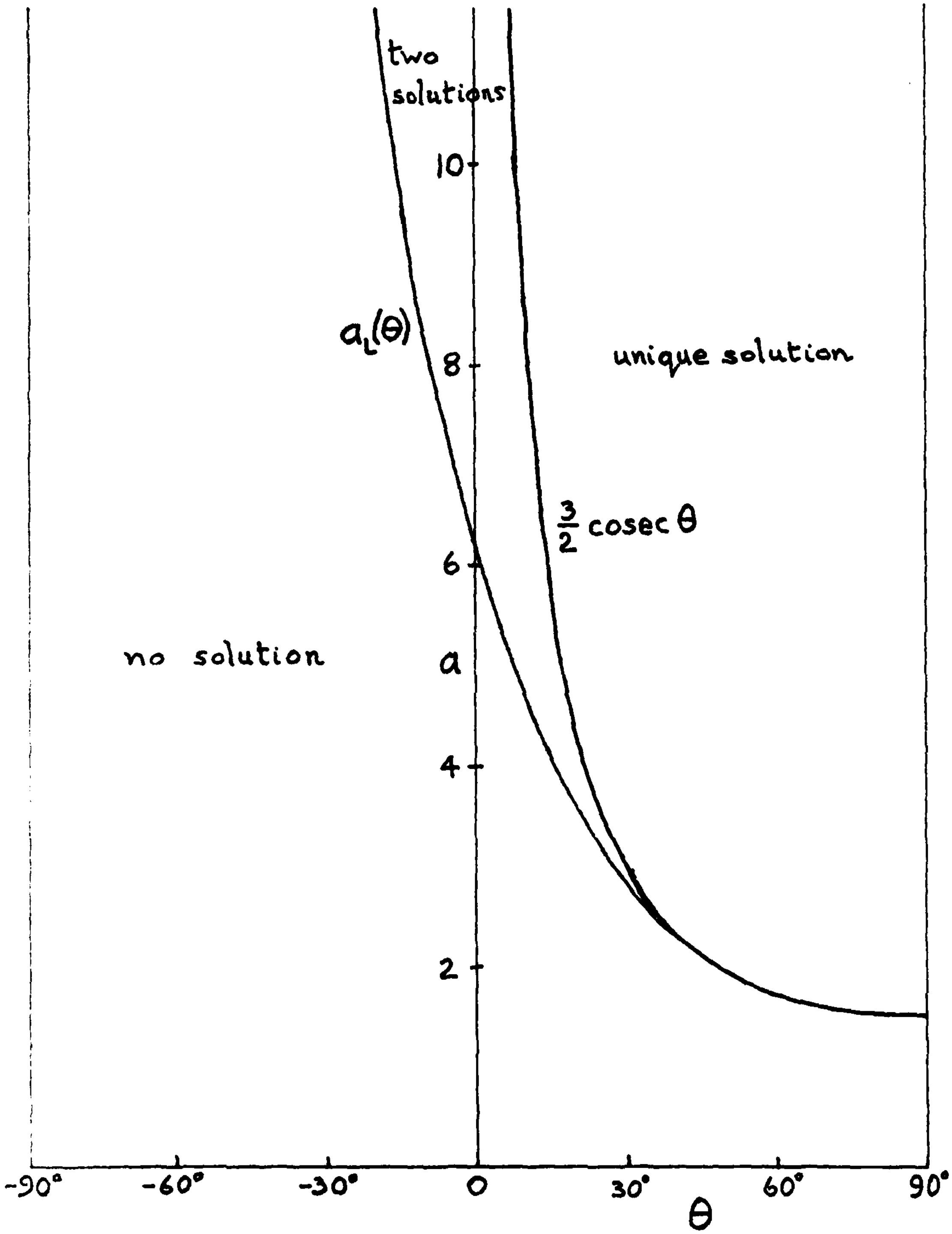

Fig 7 Values of a and θ for which upper branch solutions exist (Bryson model)

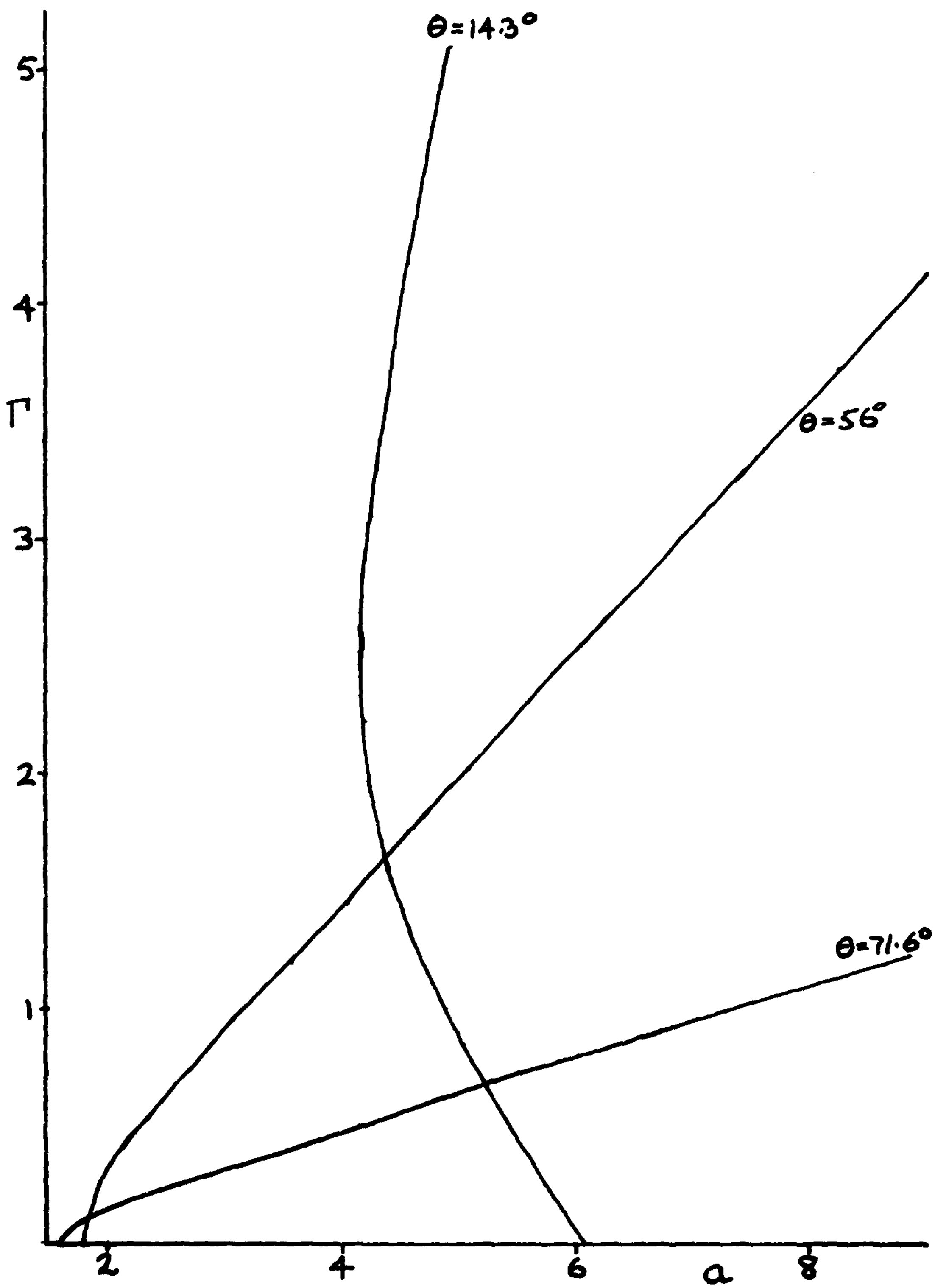

Fig 8 Variation of circulation parameter with incidence parameter, Bryson model, upper branch

Fig 9

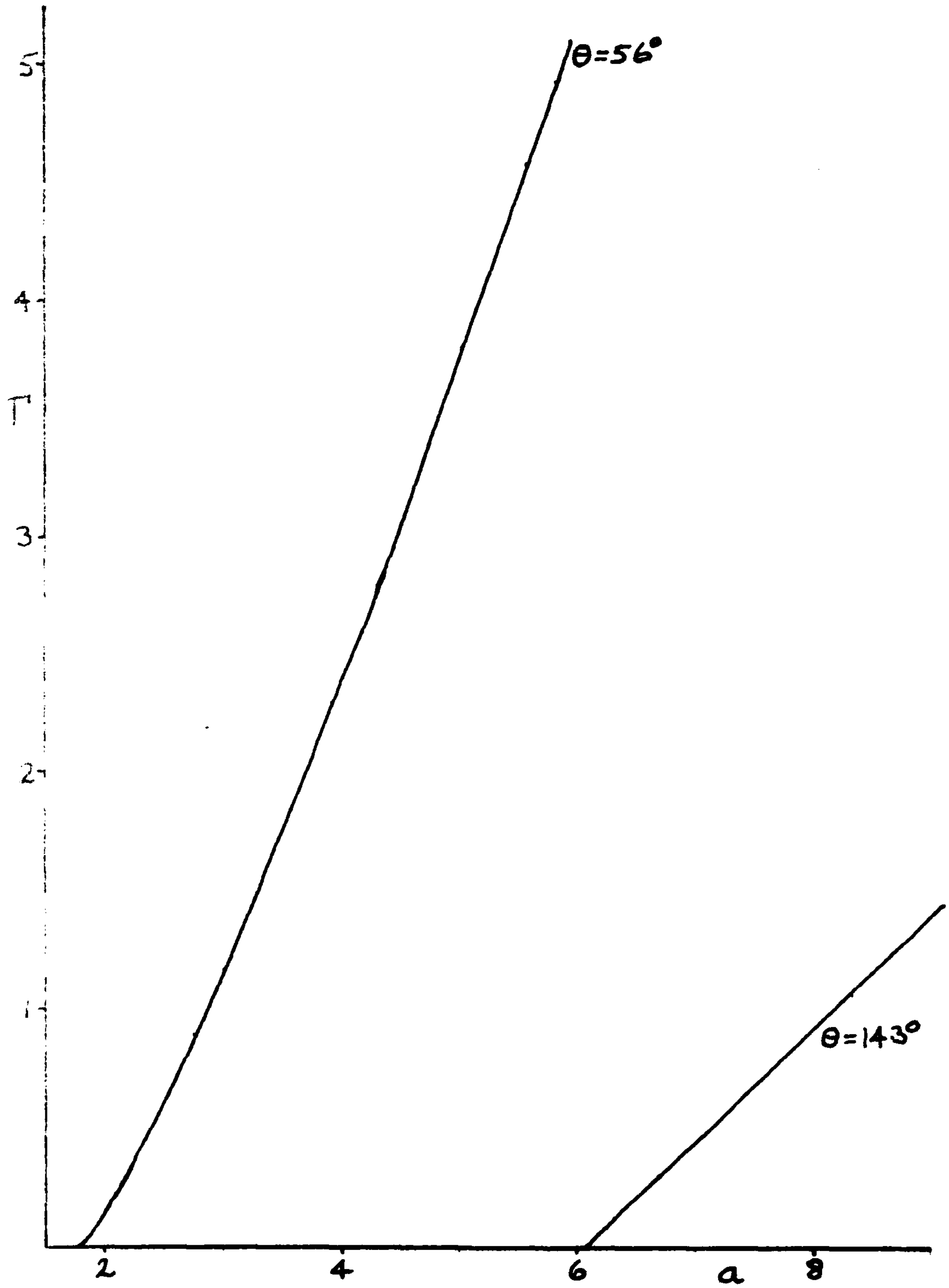

Fig 9 Variation of circulation parameter with incidence parameter, Bryson model, lower branch

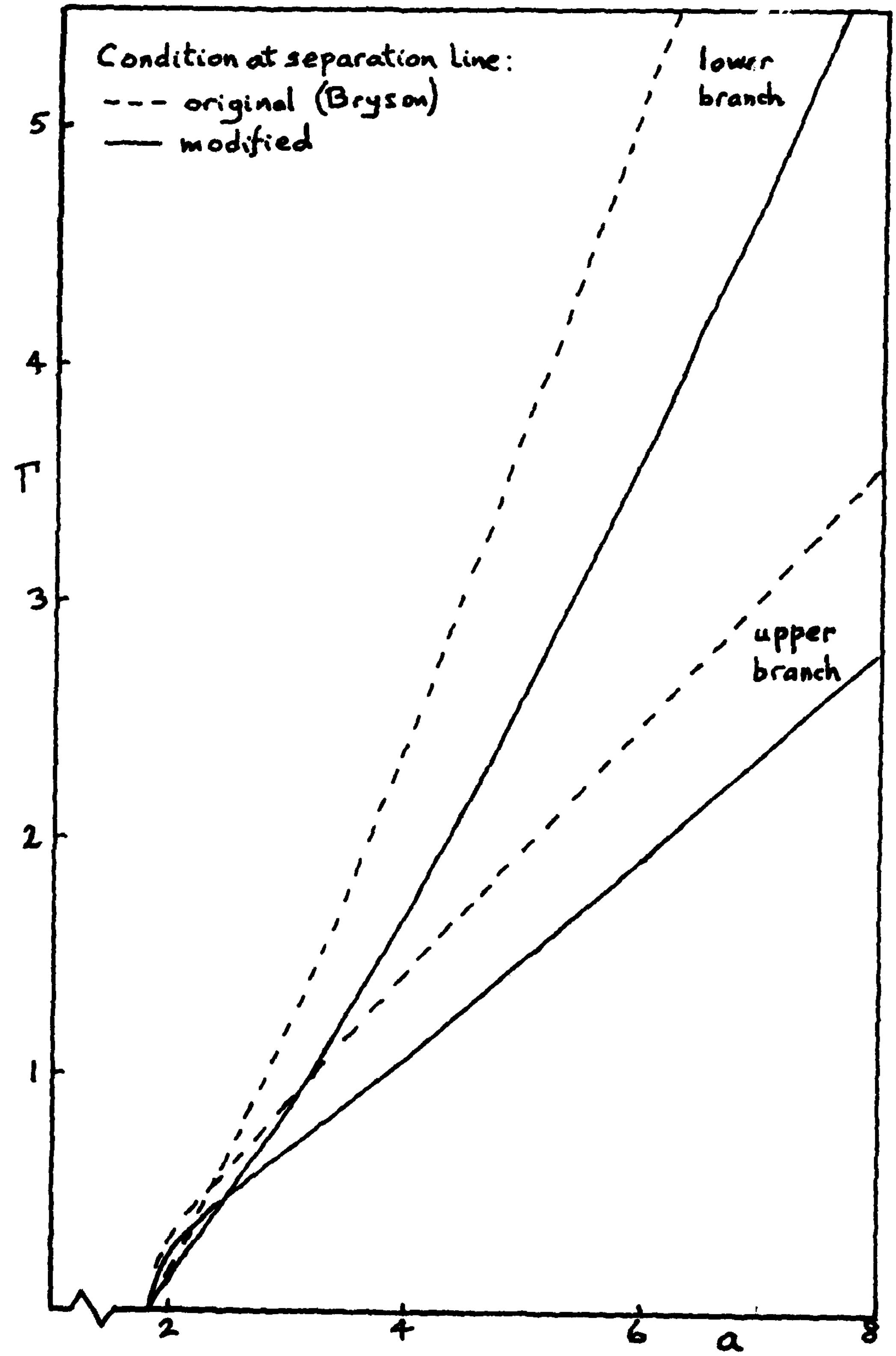

Fig 10 Variation of circulation parameter with incidence parameter for θ = 56°, both models

REPORT DOCUMENTATION PAGE

Overall security classification of this page

UNLIMITED

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17. Abstract

The simple line-vortex model of laterally-symmetric separated flow past a circular cone at incidence, originally proposed by Bryson, is studied in detail, both numerically and analytically through asymptotic expansions. Two branches of solutions are found, an 'upper' branch which includes Bryson's original results, and a lower branch, on which the solutions are found to be physically unrealistic. The physically realistic branch is shown to yield multiple solutions for separation lines to windward of a critical position. A proposed modification to the condition imposed on the flow at the separation line is shown to have little effect on the behaviour of the model.

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